International Journal of Theoretical Physics, Vol. 31, No. 1, 1992

Dirac's Conjecture about Constraints

Qi Zhi[1]

Received February 12, 1991

Combining Costa's theory with our algorithm, we recalculate Cawly's counterexample which contains χ^n-type constraints. The result shows that Dirac's conjecture holds true whether or not there are χ^n-type constraints.

1. INTRODUCTION

According to Dirac's conjecture (Dirac, 1964), all first-class constraints should be generators of gauge transformations. Costa *et al.* (1985) and Castellani (1982) proved this and derived the gauge generators of constrained systems, respectively. Moreover, Costa *et al.* (1985) calculated two examples: electrodynamics and the Christ–Lee model, and pointed out that if a quantity $f(q, p)$ is first class, it must be gauge invariant and

$$\dot{f}(q, p) \approx [f, H_T] \approx [f, H_E] \tag{1}$$

which explicitly shows that H_T and H_E generate the same time evolution for the canonical gauge-invariant functions $f(q, p)$, i.e., H_T and H_E are physically equivalent. If a quantity $f(q, p)$ is a gauge-dependent variable, H_T and H_E generate different equations of motion. For a gauge-dependent variable, it makes no sense whatsoever to compare f_T constructed in the H_T formalism with f_E constructed in the H_E formalism, because of the existence of "formalism-dependent" realizations.

In Section 2, combining the theory of Costa *et al.* (1985) with our algorithm, we recalculate Cawly's counterexample (Cawly, 1979), which was excluded by Costa *et al.* (1985) and Castellani (1982). Some final remarks and conclusions are contained in Section 3.

[1]Department of Biomedical Engineering, Capital Institute of Medicine, Beijing 100054, China.

2. RECALCULATION OF CAWLY'S COUNTEREXAMPLE

In Cawly's (1979) famous counterexample

$$L=\dot{X}\dot{Z}+1/2YZ^2 \tag{2}$$

which yields primary first-class constraints

$$\phi=P_y\approx 0 \tag{3}$$

the total Hamiltonian is

$$H_T=P_zP_x-1/2YZ^2+\xi P_y \tag{4}$$

On the basis of our previous paper (Qi, 1990), the secondary first-class constraints are

$$\chi_1=Z^2\approx 0 \tag{5a}$$

$$\chi_2=ZP_x\approx 0 \tag{5b}$$

$$\chi_3=P_x^2\approx 0 \tag{5c}$$

and within the H_T formalism the gauge generator g is

$$g=\dddot{\varepsilon}_{(t)}P_y+\ddot{\varepsilon}_{(t)}Z^2+\dot{\varepsilon}_{(t)}ZP_x+\varepsilon_{(t)}P_x^2 \tag{6}$$

Thus, the gauge transformations are

$$\delta X=\dot{\varepsilon}Z \tag{7a}$$

$$\delta Y=\dddot{\varepsilon} \tag{7b}$$

$$\delta Z=0 \tag{7c}$$

$$\delta P_x=0 \tag{7d}$$

$$\delta P_y=0 \tag{7e}$$

$$\delta P_z=-2\ddot{\varepsilon}Z-\dot{\varepsilon}P_x \tag{7f}$$

$$\delta\xi=\ddddot{\varepsilon} \tag{7g}$$

From (4), the equations of motion are

$$\dot{X}\approx P_z \tag{8a}$$

$$\dot{Y}\approx\xi \tag{8b}$$

$$\dot{Z}\approx P_x \tag{8c}$$

$$\dot{P}_x\approx 0 \tag{8d}$$

$$\dot{P}_y\approx 1/2Z^2\approx 0 \tag{8e}$$

$$\dot{P}_z\approx YZ \tag{8f}$$

which are invariant under (7).

Within the H_E formalism, the extended Hamiltonian is

$$H_E = P_z P_x + \xi_1 P_y + \xi_2 Z^2 + \xi_3 Z P_x + \xi_4 P_x^2 \tag{9}$$

and the gauge generator is

$$G = \tau_1 P_y + \tau_2 Z^2 + \tau_3 Z P_x + \tau_4 P_x^2 \tag{10}$$

and the gauge transformations are

$$\delta X = \tau_3 Z + 2\tau_4 P_x \tag{11a}$$

$$\delta Y = \tau_1 \tag{11b}$$

$$\delta Z = 0 \tag{11c}$$

$$\delta P_x = 0 \tag{11d}$$

$$\delta P_y = 0 \tag{11e}$$

$$\delta P_z = -2\tau_2 Z - \tau_3 P_x \tag{11f}$$

From (9), the equations of motion are

$$\dot{X} \approx P_z + \xi_3 Z + 2\xi_4 P_x \tag{12a}$$

$$\dot{Y} \approx \xi_1 \tag{12b}$$

$$\dot{Z} \approx P_x \tag{12c}$$

$$\dot{P}_x \approx 0 \tag{12d}$$

$$\dot{P}_y \approx 0 \tag{12e}$$

$$\dot{P}_z \approx -2\xi_2 Z - \xi_3 P_x \tag{12f}$$

which are invariant under (11), provided the Lagrange multipliers ξ_1, ξ_2, ξ_3, and ξ_4 transform, respectively, as (Costa *et al.*, 1985)

$$\delta\xi_1 = \dot{\tau}_1 \tag{13a}$$

$$\delta\xi_2 = \dot{\tau}_2 + \tfrac{1}{2}\tau_1 \tag{13b}$$

$$\delta\xi_3 = \dot{\tau}_3 + 2\tau_2 \tag{13c}$$

$$\delta\xi_4 = \dot{\tau}_4 + \tau_3 \tag{13d}$$

Analyzing (7) and (11), we find that X, P_z, and Y are gauge-dependent variables; and the basic canonical gauge-invariant functions $f(q, p)$ are Z, P_x, and P_y. So both sets of equations (8) and (12) yield the same time evolution for the gauge-invariant functions $f(q, p)$ as seen from

$$\dot{Z} \approx P_x \tag{14a}$$

$$\dot{P}_x \approx 0 \tag{14b}$$

$$\dot{P}_y \approx 0 \tag{14c}$$

This result agrees with that of Costa *et al.* (1985), but the system contains χ^n-type constraints.

3. CONCLUSION

Reexamining Costa's and Castellani's proof, we find that it is not necessary to reject χ^n-type constraints. If we do not linearize the constraints like Cawly (1979) and others. Costa's theory can contain χ^n-type constraints. This also proves indirectly the correctness of the algorithm proposed in our previous paper (Qi, 1990). We conclude the Dirac's conjecture holds true whether or not there are χ^n-type constraints.

ACKNOWLEDGMENT

I would like to thank Prof. Li Zi-Ping for his immense help and valuable discussion.

REFERENCES

Castellani, L. (1982). *Annals of Physics*, **143**, 357.
Cawly, R. (1979). *Physical Review Letters*, **42**, 413.
Costa, M. E. V., Girotti, H. O., and Simoes, T. J. M. (1985). *Physical Review D*, **32**, 405.
Dirac, P. A. M. (1964). *Lecture on Quantum Mechanics*, Belfer Graduate School of Science, Yeshiva University, New York.
Qi Zhi. (1990). *International Journal of Theoretical Physics*, **29**, 1309.